The AP Physics Summer Assignment: Calculus

This is a quick exposure to the Calculus that you will need next year in AP Physics. We will start using derivatives on the first day and integration and differential equations after three weeks. This is not a course in Calculus, which your math teacher will provide (or has provided) more completely and with greater clarity and nuance. All of this instruction will be repeated as each of these ideas is needed by Physics, but I think it will be helpful if that is a reminder instead of your first exposure. AP Physics is an extremely demanding course and if you are also learning calculus at the same time, it can be overwhelming. But take heart, others have gone before you and flourished and been happy.

You may find this frustrating and angering and maybe even disheartening sometimes, but don't let it be defeating. Walk away from it and then come back and try again. Reach out to Mr. Houghton at hhoughton@gsgis.k12.va.us with your questions. Try Khan Institute. Try your classmates. Ultimately, even if you don't feel that you have gained mastery from this experience, you will be in a stronger, more prepared place when you encounter this material again next Fall.

Some of you have already been through all of this Calculus and should find these ideas and exercises easy. Good for you. If you want to improve upon this document with editing and rewriting, then we would both benefit.

When you come to the first day of class, each of you will have a document with the lettered exercises **legibly** worked. Unless you are going to have a document saying that the AP graders do not have to be able to read what you write, you are going to have to start working on sharing your problem solving cogently and legibly right now. It is fine if you work a problem and then discover that you got the wrong answer and then start again (and again) until you succeed… let me see it all… legibly and completely. I would encourage you not to look at answers until you have first tried to do the problem. To the extent you invest yourself in the problem solving, you will reap the rewards of intellectual growth.

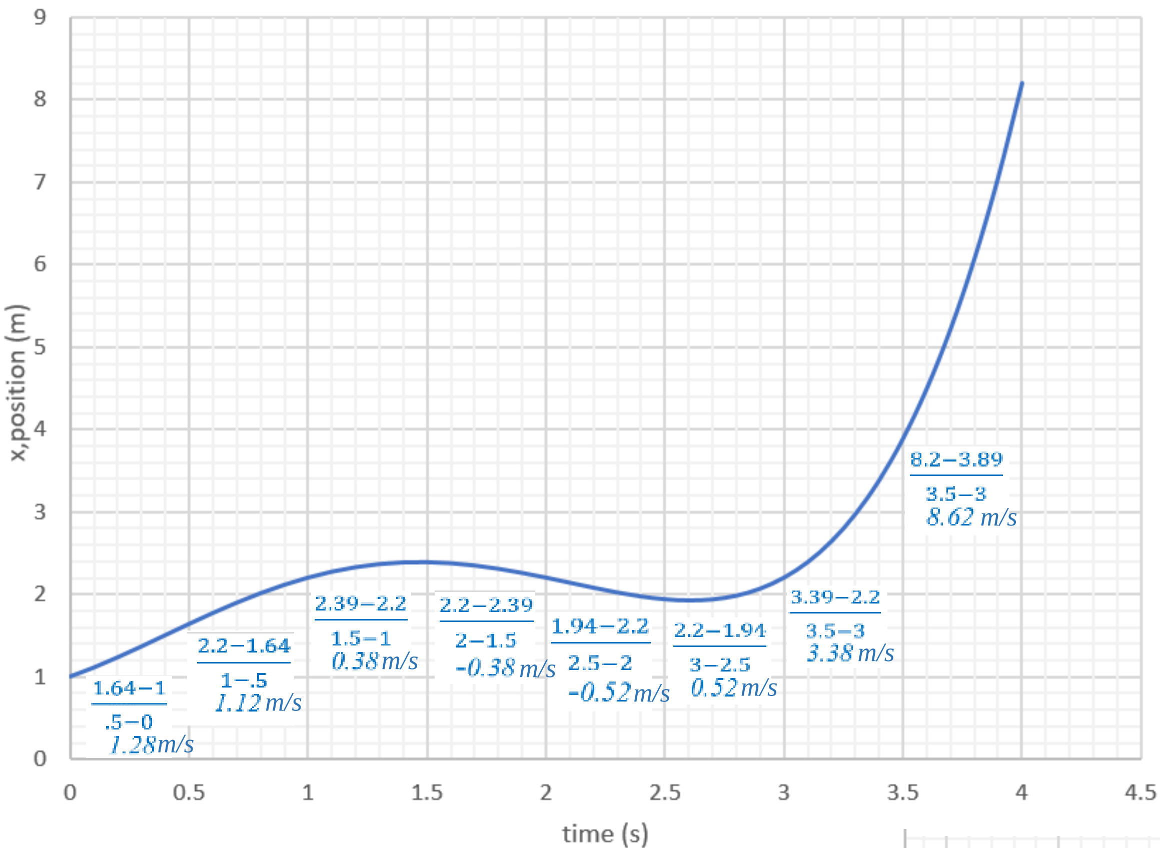
I am asking a lot. If you feel it is too much, I will give you full credit for only doing every other problem. However, if you are experiencing these ideas for the first time, the more practice you have with them, the more you will gain. Don't be too distressed if you still don't feel confident. It took me three tries at Calculus before I felt I fully understood the ideas… perhaps in the Fall it will all come together… because of the time you invest now.

Bon Appetit
Mr. Houghton

Calculus for AP Physics

Newton invented calculus (*at about the same time as Leibnitz*) to deal with CONTINUOUS VARIATION. He was trying to describe the Moon's motion as it orbited the earth and from moment to moment, the Moon would be in a new location, going in a new direction. The force from the earth changed magnitude with each new distance and the direction of the force changed and this changed the way the Moon moving.

Consider this graph of an object's position as a function of time. As indicated, the rate of change in position ($\Delta x/\Delta t = v_{average}$) is different in each subsequent half second of the graph. This is *continuous variation*.



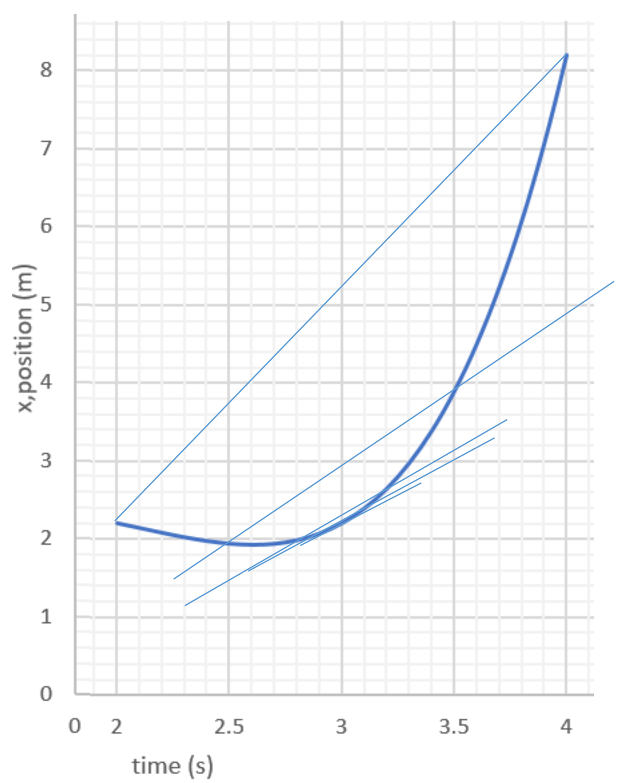
Beneath each interval $\Delta x/\Delta t$, which is the same as $(x_f-x_i)/(t_f-t_i)$, is shown and then the value of the average velocity for the interval.

But what is the velocity at EXACTLY 3 seconds?

Observe the values of the average velocity as the intervals grow ever closer to 3 seconds.

from 2 seconds to 4 seconds: 3.0000 m/s
from 2.5 to 3.5 : 1.9500 m/s
from 2.8 to 3.2 : 1.6560 m/s
from 2.9 to 3.1 : 1.6140 m/s
from 2.95 to 3.05 : 1.6035 m/s
from 2.99 to 3.01 : 1.6001 m/s not drawn
from 2.999 to 3.001 : 1.6000 m/s not drawn

The average velocities are approaching a "limiting value" of 1.6 m/s. We say the "instantaneous velocity" at 3 seconds is 1.6 m/s.



Average velocity = $\Delta x/\Delta t$

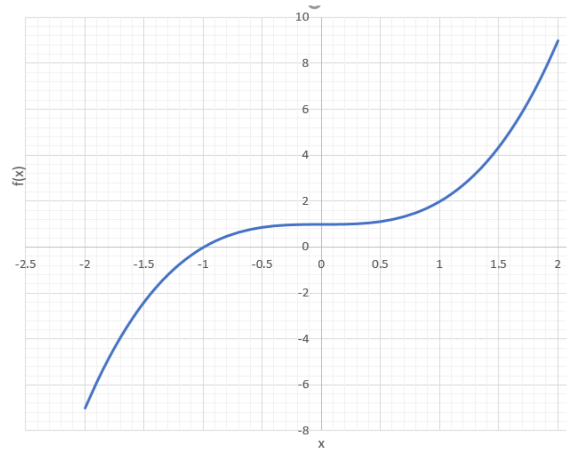
If there is a limiting value as Δt becomes *infinitesimally* small, then $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ = instantaneous velocity = v without subscript.

DERIVATIVES

If f(x) is a function of x and the rate of change in f(x) with respect to x approaches a limiting value as the change in x becomes infinitesimal (vanishingly small), then we say

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx}$$ = derivative of f(x) with respect to x note how our symbols change from the clunky Greek Δ for a gross change to the Latin d for infinitesimal change.

We are now going to work through the mechanics of evaluating derivatives, but we need to always come back to the meaning of "**instantaneous rate of change**" as illustrated on the previous page.



Recall that Δx = change in x = $x_{final} - x_{initial}$

The graph at the right is $f(x) = x^3 + 1$

If we apply the definition $\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \frac{df(x)}{dx}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx}$$

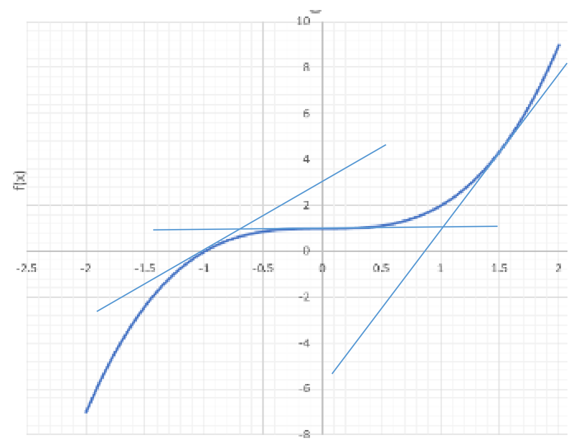
f(x) = x³ +1: $$\lim_{\Delta x \to 0} \frac{((x+\Delta x)^3+1) - (x^3+1)}{\Delta x} = \frac{df(x)}{dx}$$

$$\lim_{\Delta x \to 0} \frac{(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3+1) - (x^3+1)}{\Delta x} = \frac{df(x)}{dx}$$

subtract (x^3 +1): $$\lim_{\Delta x \to 0} \frac{(\quad 3x^2\Delta x+3x\Delta x^2+\Delta x^3 \quad)}{\Delta x} = \frac{df(x)}{dx}$$

Δx divides into top: $$\lim_{\Delta x \to 0} \frac{(\quad 3x^2+3x\Delta x^1+\Delta x^2 \quad)}{1} = \frac{df(x)}{dx}$$

Δx→0 means $3x\Delta x^1+\Delta x^2 \to 0$ $$3x^2 = \frac{df(x)}{dx}$$



And thus we have determined that the instantaneous rate of change of x^3 +1 (with respect to x) is $3x^2$.

In a graph, this rate of change is the slope.   You ought to convince yourself that in the graph, the slope at x= -1 is $3(-1)^2 = 3$.

The slope at x= 0 is $3(0)^2 = 0$.   The slope at x= 1.5 is $3(1.5)^2 = 6.75$.

In physics, where we use x specifically to mean position,

$\frac{\Delta x}{\Delta t} = average\ velocity,\quad \frac{dx}{dt} = instantaneous\ velocity.$

You will want to always remind yourself of the graphical meaning of derivative of position with respect to time we have just developed: the rate change in position for a very small change of time.

RULE 1: $\frac{dx^b}{dx} = bx^{b-1}$

Try to apply this rule to these derivatives and check your answers at the top of the next page

A. $\frac{dx^4}{dx} =$

B. $\frac{dt^{-1}}{dt} =$ t's exponent is -1

C. $\frac{dy^{\frac{1}{3}}}{dy} =$ y's exponent is 1/3rd

A. $4x^3$ B. $-t^{-2}$ or $\frac{-1}{t^2}$ C. $\frac{1}{3}y^{\frac{-2}{3}}$ or $\frac{1}{3\sqrt[3]{y^2}}$

RULE 2: $\frac{d(mx^b)}{dx} = mbx^{b-1}$

For example, $\frac{d(6k^5)}{dk} = 30k^4$

RULE 3: $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ where u and v are functions of x

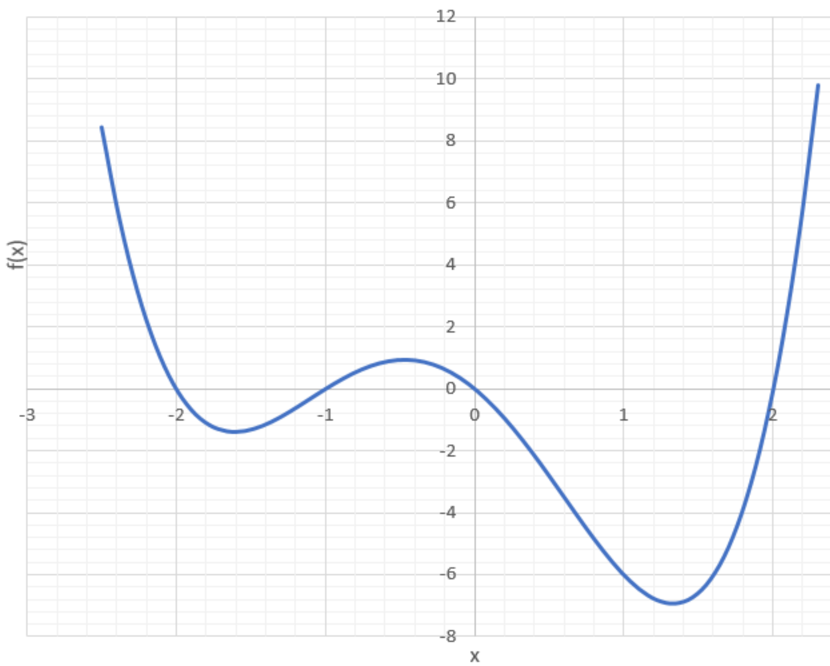
For example, $\frac{d(6b^5+3b^2)}{db} = 30b^4 + 6b$

Try to apply these rules to these derivatives and check your answers at the top of the next page

D. $\frac{d(3t^4-2t^3+t+7)}{dt} =$ it might be helpful to note that $t + 7 = t^1 + 7t^0$

E. $\frac{d(8x^2-2x+3x^{-2})}{dx} =$

EXTREMA



Frequently in physics, you will be asked to find when a function is maximum or minimum. Consider the function graphed at the right. There is a local minimum at -1.6, a local maximum at -0.44 and a minimum at 1.3. What is happening to the slope of the graph at these extrema? Necessarily, the slope is changing between positive and negative and must pass through a zero.

Whenever you encounter a problem where you are asked for the smallest or the fastest, you should think about how that would appear in a graph and use the derivative to find when it happens and then substitute that value into the function to see how big it is.

A ball thrown into the air has its vertical position described by $y = 98t - 4.9t^2$. When is the ball as its highest point and how high is it then? You could try this problem if you are courageous, or you could study the answer at the top of the next page. After studying the answer, return here and see if you are able to reproduce that work... not from memory, but because you have internalized the process.

D. $12t^3 - 6t^2 + 1$ E. $16x - 2 - 6x^{-3}$

We are asked to find the maximum y where $y = 98t - 4.9t^2$. If we graphed y vs. t we would see a parabola and the largest y value would be where dy/dt = 0. dy/dt = 98 – 9.8t and dy/dt =0 when t = 10 seconds. If we put 10 in for t in $y = 98t - 4.9t^2$, y = 980 – 490 = 490 meters.

Here is a problem very similar to one you will be given on the first day of class in the Fall:

F. The position of a particle moving on the x-axis is given by $x = 12t^2 - 2t^3$ where x is in meters and t is in seconds. Determine a) the position and velocity of the particle at t = 3.0 s.
b) What is the maximum positive coordinate reached by the particle and at what time was it reached?
c) What is the maximum positive velocity reached by the particle and at what time was it reached?

And another problem for more practice

G. A particular bucket has water flowing into it at an ever-decreasing rate. The bucket also has a hole in its bottom that is continuously getting bigger. At time zero there is one liter of water in the bucket. The water coming in is described by the equation $w_i = 9t^2$. The water leaving via the hole is described by the equation $w_o = 2t^3 + 12t$.
The equation describing the amount of water in the bucket is $w_{bucket} = -2t^3 + 9t^2 - 12t + 8$. The water in the bucket will initially decrease to a least amount before increasing slightly and then all the water will run out. What is the least water in the bucket during the initial decrease and when will it occur?

SPECIAL DERIVATIVES

It is important that you know these… and that is enough. I also share the derivation of dsinθ/dθ in case you are curious.

dSIN(x)/dx = COS(x)

recall that sin(a+b) = sin(a)cos(b) + cos(a)sin(b)

$$\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} = \frac{d\sin(x)}{dx}$$

$$\lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} = \frac{df(x)}{dx}$$ but as Δx→0, sin(Δx) → Δx and cos(Δx) → 1

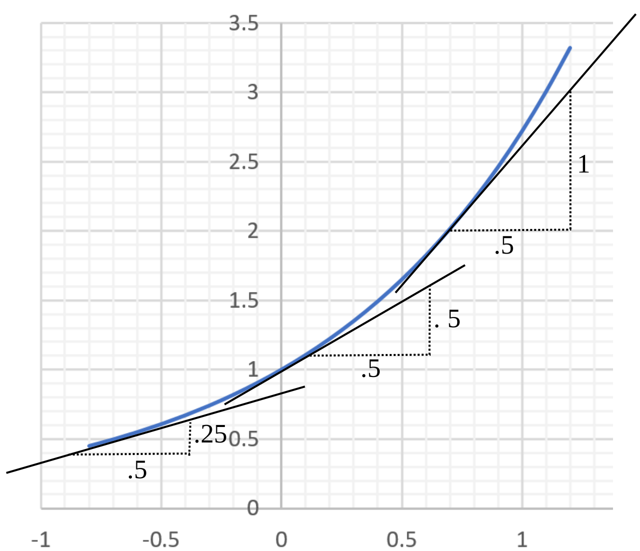
$$\lim_{\Delta x \to 0} \frac{\sin(x)\, 1 + \cos(x)\,\Delta x - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x)\,\Delta x}{\Delta x} = \cos(x)$$

dCOS(x)/dx = -SIN(x)

recall that cos(a+b) = sin(a)sin(b) - cos(a)cos(b) and then do a similar development to above

d(e^x)/dx = e^x

this can be proven but not so easily. Note that what this says is that in the graph of e^x versus x at the right, the slope of the graph at f(x) = e^x is d(f(x)/dx = e^x.
When e^x = ½, the slope is ½.
When e^x = 1, the slope is 1.
When e^x = 2, the slope is 2.

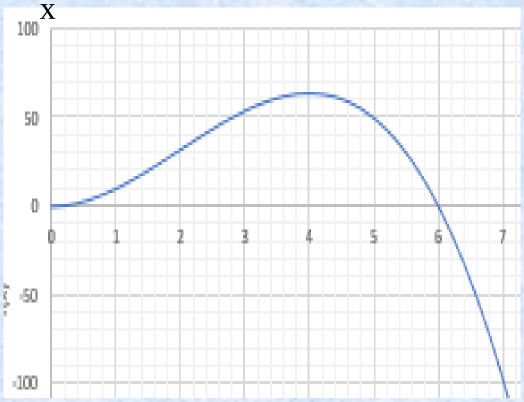


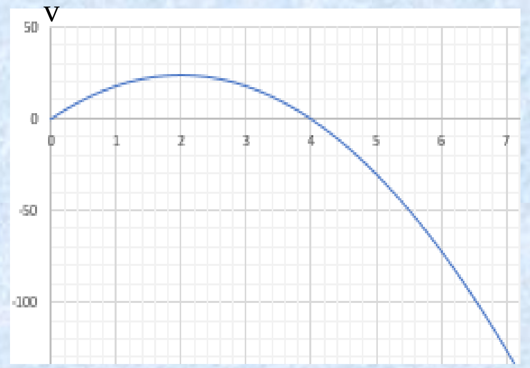
F. a) $x = 12(3^2) - 2(3^3) =$ **54 m**, $v = dx/dt = 24(3) - 6(3^2) =$ **18 m/s**

b) x is at an extreme when $dx/dt = 0$, $24t - 6t^2 = 0$ when **t = 4s** @ $t = 4$, $x = 12(4^2) - 2(4^3) =$ **64 m**

c) v is at an extreme when $dv/dt = 0$, $24 - 12t = 0$ when **t = 2s** @ $t = 2$, $v = 24(2) - 6(2^2) =$ **24 m/s**

See if these graphs of x vs t and v vs t do not confirm your results.





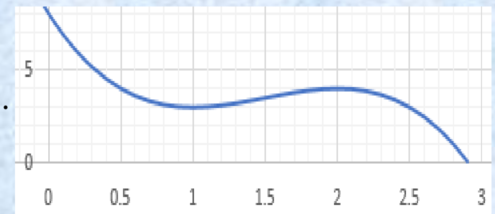
G. $w_{bucket} = -2t^3 + 9t^2 - 12t + 8$ is continuously varying. It will be a maximum or minimum when $dw/dt = 0$.

$dw/dt = -6t^2 + 18t - 12 = 6(-t + 1)(t - 2)$ which equals zero when $t = 1$ or 2.

Substituting 1 and 2 into $w_{bucket} = -2t^3 + 9t^2 - 12t + 8$ yields

$w_{bucket} = 3$ gallons @ $t = 1$s and $w_{bucket} = 4$ gallons @ $t = 2$s

Answer is **3 gallons @ t = 1**s Compare this with the graph at the right.



PRODUCT RULE

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$ where u and v are functions of x

eg1: $$\frac{d(x^4x^6)}{dx} = \frac{d(x^{10})}{dx} = 10x^9$$ **not** using product rule

$$\frac{d(x^4x^6)}{dx} = x^4(6x^5) + x^6(4x^3) = 6x^9 + 4x^9 = 10x^9$$ so you can see that the product rule does work!

eg2: $$\frac{d(\frac{x^4}{x^6})}{dx} = \frac{d(x^{-2})}{dx} = -2x^{-3}$$ **not** using product rule

as product: $$\frac{d(x^4x^{-6})}{dx} = x^4(-6x^{-7}) + x^{-6}(4x^3) = -6x^{-3} + 4x^{-3} = -2x^{-3}$$

as a quotient: here is a ditty: "low d-high minus high d-low, draw the line and square below"

$$\frac{d(\frac{x^4}{x^6})}{dx} = \frac{x^6(4x^3) - x^4(6x^5)}{x^{12}} = \frac{4x^9 - 6x^9}{x^{12}} = -2x^{-3}$$

low d-high – high d-low, draw the line and square below

eg3: $$\frac{\boldsymbol{d\,(tanx)}}{\boldsymbol{dx}} = \frac{d\,(sinx/cosx)}{dx} = \frac{cosx\frac{d\,(sinx)}{dx} - sinx\frac{d\,(cosx)}{dx}}{cos^2x} = (cos^2x + sin^2x)/cos^2x = \boldsymbol{sec^2x}$$

Now you practice. Answers at top of next page

H. $$\frac{d\,(5p^4p^2)}{dp}$$ do not simplify first

I. $$\frac{d\,(3x^2cosx)}{dx}$$

H. $20p^3p^2 + 10p^4p = 30p^5$ I. $6x\cos x - 3x^2\sin x$

CHAIN RULE

Derivative of a function of a function. You must be vigilant to recognize when this is the case.

$\frac{d(u^b)}{dx} = bu^{b-1}\frac{du}{dx}$ *(where u is a function of x)* after doing the normal derivative thing with the exponents as if you were taking derivative with respect to "u", be careful to take derivative of "u" with respect to "x".

or $\frac{d(g(f))}{dx} = \frac{dg}{df}\frac{df}{dx}$ *(where f is a function of x and g is a function of f)*

eg1: $\frac{d\,(sin^3x)}{dx} = 3sin^2x\frac{dsinx}{dx} = 3sin^2xcosx$

eg2: $\frac{d\,(7x^2y^3)}{dt} = 14xy^3\frac{dx}{dt} + 21x^2y^2\frac{dy}{dt}$

Now you practice. Answers at bottom of next page

J. $\frac{d\,(5cos^4g)}{dg}$

K. $\frac{d\,(3e^{2sinx})}{dx}$

L. $\frac{d\,((y^3-1)^4(y^2-8)^5}{dy}$ leave answer unsimplified

M. $\frac{d\,(3a^2b^3)}{dw}$ *where a and b are functions of w*

PHYSICS PROBLEMS USING CALCULUS (answers and working on next page… don't go there too quickly)

N. A particle moves so that its position (in meters) as a function of time (in seconds) is r(t) = 7(in x direction) + $5t^3$ (in y direction) + t (in the z direction). Write expressions for its velocity and acceleration as functions of time. Recall that v = dx/dt and a = dv/dt

O. A particle leaves the origin with initial velocity, v = 7 m/s and constant acceleration, a = -2 m/s/s. Using the equation $x = x_i + v_it + ½ at^2$, find what the particle's position and velocity are when it reaches its maximum x coordinate.

P. Ship A's position, r = (6 + 2t)(x-direct) + (3 – 5t)(y-direct) in kilometers and minutes.
Ship B's position, r = (1 + 3t)(x-direct) + (2 + t)(y-direct).
The distance^2 between the ships is $r_{x-direct}^2 + r_{y-direct}^2$ which I think is easier to take the derivative of instead of distance = $\sqrt{r_{x-direct}^2 + r_{y-direct}^2}$. When is the distance between the ships least and what is that distance?

Q. Force = ma. If F = $2x^3 - 3x^2 - 12x + 4$ and m = 1, at what position will the velocity be changing most rapidly? You will find two extrema and must choose which corresponds to a maximum.

J. $-20\cos^3 g(\sin g)$ K. $6(\cos x)e^{2\sin x}$

L. $(4(y^3-1)^3(3y^2))(y^2-8)^5 + (y^3-1)^4(5(y^2-8)^4(2y))$ M. $6ab^3(da/dw) + 9a^2b^2(db/dw)$

N. v = d(r(t))/dt = d(7(x direct) + $5t^3$ (y direct) + t (z direct))/dt = **$15t^2$(y direct) + 1(z direct)**

a = dv/dt = d($15t^2$(y direct) + 1(z direct))/dt = **30t (y direct)**

O. x is max when dx/dt = 0 d $(0 + 7t + ½ (-2)t^2)$/dt = 7 – 2t which equals zero when t = 3.5 s

At 3.5 s, **x** = $0 + 7(3.5) – (3.5)^2$ = **12.25 m**

At 3.5 s, **v =** dx/dt =7 – 2t = 7 – 2(3.5) **= 0 m/s** when at max x you have stopped moving away!

P. Dist is least when $dist^2$ is least and this is when d($dist^2$)/dt = 0.

$d(\{(6+2t)-(1+3t)\}^2 + \{(3-5t)-(2+t)\}^2)/dt = d(\{5-t\}^2 + \{1-6t\}^2)/dt = 2(5-t)(-1) + 2(1-6t)(-6) = -22+74t$

= 0 if t **= 0.297 min**

At which time dist = $\sqrt{(5-0.297)^2 + (1-0.297)^2} = \mathbf{4.77\ km}$

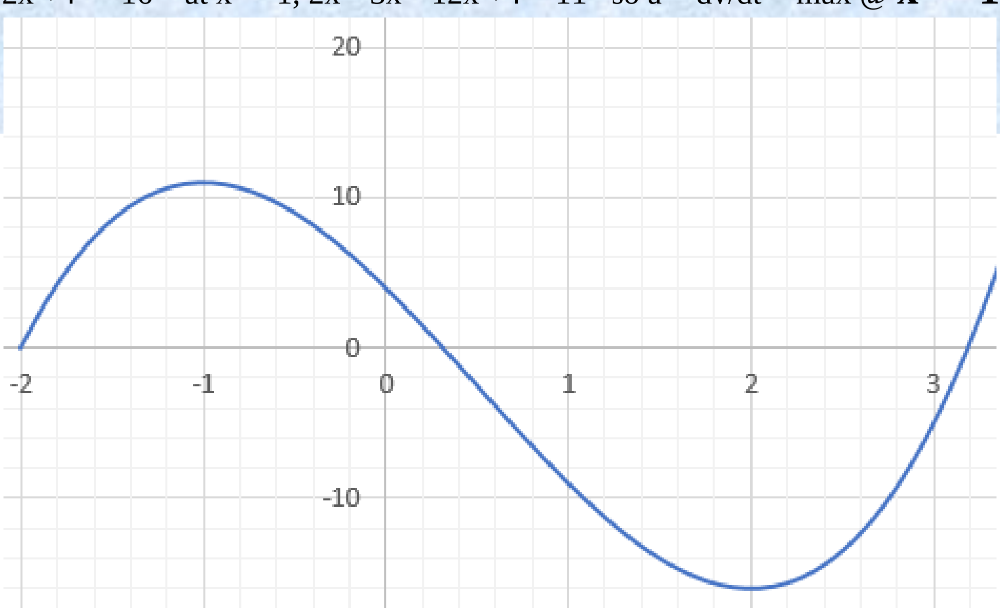
Q. a = F/m = $2x^3 -3x^2 -12x +4$ = dv/dt dv/dt max when acceleration, a is maximum.

Think of graphing a vs x…when a is greatest, da/dx= 0

d($2x^3 -3x^2 -12x +4$)/dx = $6x^2 – 6x -12 = 6(x-2)(x+1)$ which equals zero when x = 2 or -1

at x = 2, $2x^3 -3x^2 -12x +4 = -16$ at x = -1, $2x^3 -3x^2 -12x +4 = 11$ so a = dv/dt = max @ **x = -1**

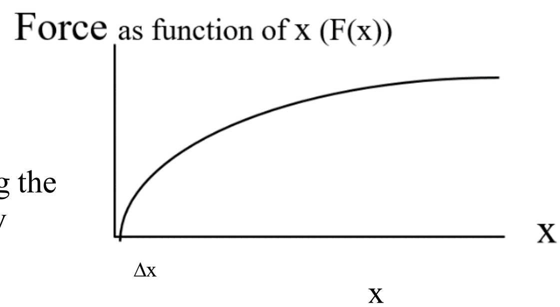
observe this graph

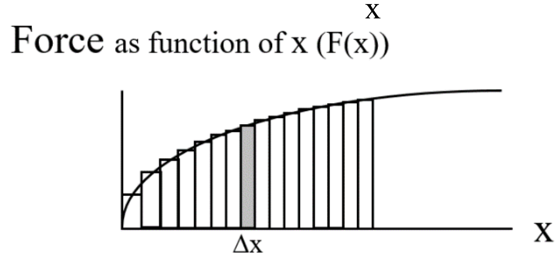


INTEGRALS

Again, in the context of **continuous change**, the integral represents a continuous summation of addends that are continuously changing. The symbol for an integral is ∫ which is just a stretched out "s" standing for summation. The other symbol for summation is Σ. This will be referred to as "discrete" summation (chunks) as opposed to continuous. The idea of **continuous summation** should guide your use of integration in physics, but the evaluation of integrals will be accomplished by thinking of them as **anti-derivatives.**

FINDING WORK: an example





You may recall that work done by a force equals the force times the distance through which the force acts.

Work = Force · distance

But suppose the force varies with position.

You could approximate the work (area under curve) by cutting the region into little rectangles whose height is the approximately constant force at that location and whose width is a small Δx.

Work = ∑ F(x) Δx

In the limit as the width of the rectangles become infinitesimally narrow, this becomes a continuous sum. The discrete summation symbol, ∑ is replaced with a stretched s, ∫, and Δx becomes dx.

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x = \int_a^b f(x)dx$$

ANTI-DERIVATIVES

The actual evaluation of integrals will be a matter of figuring out what you would have taken the derivative of to get the argument of the integral: $if\ \frac{dx}{dt} = v\ then\ \int v dt = x + C$

(note that if C is a constant then dC/dt = 0 and d(x + C)/dt = dx/dt + dC/dt = v)

RULES

1. $\int t^b\, dt = \frac{1}{b+1} t^{b+1} + C$

2. Where u and v are functions of x

a. $\int au\, dx = a\int u\, dx$

b. $\int (u + v)\, dx = \int u\, dx + \int v\, dx$

Try these problems and the check the answers at the top of the next page

R. $\int 4z\, dz$ S. $\int x^2\, dx$ T. $\int 1/y^3\, dy$ U. $\int 5m^3 + 4m^2\, dm$ V. $\int \frac{1}{t^4} + \frac{3}{t^2}\, dt$

R. $2z^2 + C$ S. $\frac{x^3}{3} + C$ T. $\frac{-1}{2\,x^2} + C$ ($\int y^{-3}\,dy = \frac{1}{(-3+1)} y^{(-3+1)} + C$)

U. $\frac{5m^4}{4} + \frac{4m^3}{3} + C$ V. $\frac{-\,t^{-3}}{3} - 3t^{-1} + C = \frac{-1}{3\,t^3} - \frac{1}{3t} + C$

Here are some more integrals. This can be a chance for you to recall the special derivatives you learned earlier. Answers at top of next page.

W. $\int \cos\theta d\theta$ ask yourself; "what do I take the derivative of, to get cosθ? then remember to add a C *(constant of integration)*.

X. $\int e^y\,dy$

Y. $\int \sqrt{p}\,dp$

Z. $\int \sin m\,dm$

And now a very special integral that will come up regularly.

$\int \frac{dx}{x} = \int x^{-1}\,dx = \frac{1}{-1+1} x^{(-1+1)} + C$ x^0 is fine but $\frac{1}{0}$ is not!

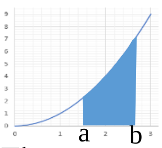
It can be derived, but just trust me…

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{x}} = \int x^{-1}\,dx = \mathbf{ln}\,\boldsymbol{x}$$

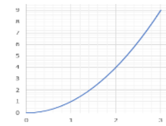
W. sinθ + C X. e^y + C Y. $\frac{2}{3}p^{1.5}$+ C ($\int p^{.5}\,dp = \frac{1}{(.5+1)}y^{(.5+1)} + C$)

Z. -cosθ + C

DEFINITE INTEGRAL



Up until now we've been dealing with "indefinite integrals."
If we think of an integral as finding the area under a function,

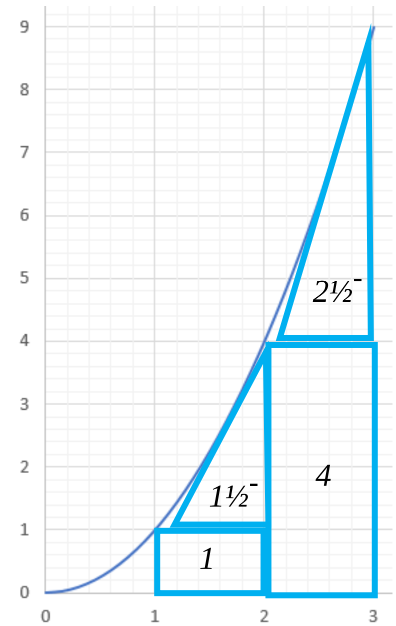


a definite integral finds the area between specified limits (a and b in this image).

The two graphs above are y = x^2 The definite integral is written $\int_a^b x^2\ dx$ which reads "the integral from a to b of x^2 with respect of x." When we evaluate it, I like to write $\left(\frac{1}{3}x^3 + C\right)\Big|_a^b = \frac{1}{3}b^3 - \frac{1}{3}a^3$

For example, find the area under the function y = x^2 from x = 1 to 3.

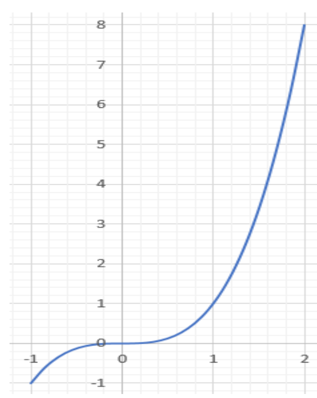
$$\int_1^3 x^2dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}3^3 - \frac{1}{3}1^3 = 9 - \frac{1}{3} = 8.67$$



note: the constant of integration is no longer written

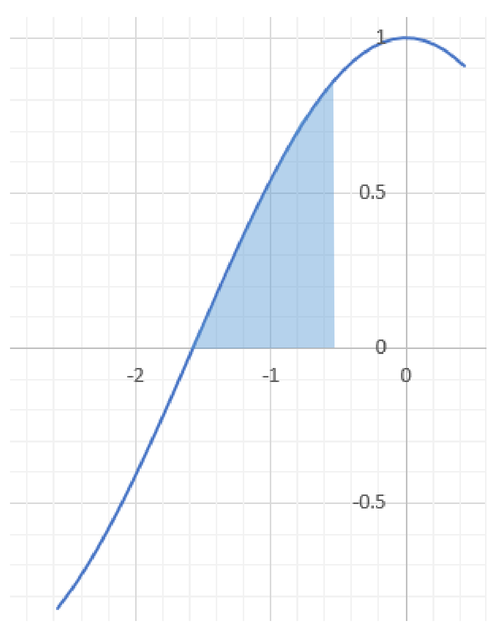
see how the areas in the boxes and triangles at the right do come close to 8.67

A. evaluate $\int_{-1}^{2} x^3dx$ then compare your result with this graph



B. $\int_{-\pi/2}^{-\pi/6} cos\beta d\beta$

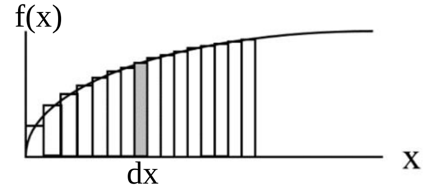
then compare your result with this graph



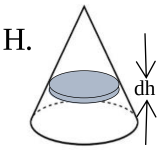
A. 3.75 B. 0.5

INTEGRALS AS CONTINUOUS SUMS

In physics, it will frequently be the case that the integral is useful as a continuous sum (again the symbol ∫ is an s for summation). Often, it is said that derivatives are the instantaneous slope of a function and integrals are the area under the function, but the reason integrals are the area, is that we are ADDING up all the infinitesimals of area f(x)xdx as shown in the graph at the right.

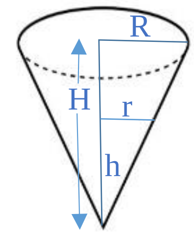


Consider this approach to finding the volume of a cone with a circular base of radius R and a height H. Cut the cone into infinitesimals of volume which are disks of radius r and thickness dh.



$$Volume = \int_0^H (\pi r^2)dh$$

But to evaluate the integral we need to figure out how r is a function of what h we have risen to. When h = 0 at base, r = R. When h = H at point, r = 0. Just because it makes it easier, let's turn the cone to have its point at the bottom. r/h = R/H so r = hR/H

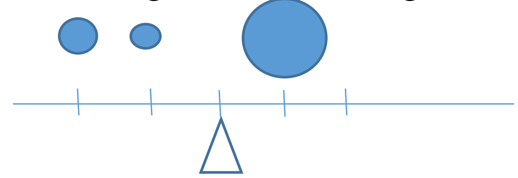


$$Volume = \int_0^H \left(\pi\left(\frac{hR}{H}\right)^2\right)dh = \int_0^H \pi\frac{h^2R^2}{H^2}dh = \frac{1}{3}\pi\frac{h^3R^2}{H^2}\Bigg|_0^H = \tfrac{1}{3}\pi HR^2 - 0$$

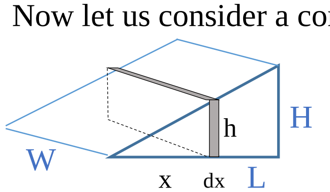
Which is hopefully a familiar result.

One of the problems we will deal with about 1/3rd of the way through the Fall is finding center of mass. The approach is to "weight" each location with how much mass is located there. Center of mass is the sum of all the weighted locations divided by the total mass. Eg: if 4 kg is at x = 2 m, 3 kg is at x = 4 m, 11 kg is at x = 8 m, then (4x2 + 3x4 + 11x8)/(4 + 3 + 11) = 108/18 = 6 m

I hope the result seems at least possible.



Now let us consider a continuous distribution.. a wedge of height H, length L and width W.



Where along its length is is its center of mass? This is going to get confusing.

Follow each step carefully. Center of mass = $\dfrac{\int xdm}{M}$ where we have broken the wedge into rectangular infinitesimals of mass and multiplied each by its location, but to evaluate the integral we would need to know x as a function of m… yikes!

If the infinitesimal drawn above is at x, then its height is x(H/L) since h/x = H/L in similar triangles.

If uniform density, density $= \dfrac{\boldsymbol{M}}{\boldsymbol{V}} = \dfrac{dm}{dV} = \dfrac{dm}{hWdx} = \dfrac{\boldsymbol{dm}}{\boldsymbol{\frac{xH}{L}Wdx}}$.

Multiplying the last denominator times $\dfrac{\boldsymbol{M}}{\boldsymbol{V}}$ $yields\ that\ dm = \dfrac{M}{V}\dfrac{xH}{L}Wdx = \dfrac{M\frac{xH}{L}Wdx}{\frac{1}{2}HLW} = \dfrac{2Mxdx}{L^2}$

Using this in the integral above: Center of mass = $\dfrac{\int_0^L x\left(\frac{2Mxdx}{L^2}\right)}{M} = \int_0^L \dfrac{2x^2dx}{L^2} = \dfrac{2x^3}{3L^2}\Bigg|_0^L = \dfrac{2L^3}{3L^2} - 0 = \dfrac{2L}{3}$

While you have no basis for saying the answer is correct, at least it is not unreasonable.

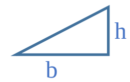
We have solved the problem by breaking it into little pieces (infinitesimals) and adding them up.

Always we need to be evaluating an INTEGRAL of a FUNCTION-OF-X times **dX** (where x is any variable).

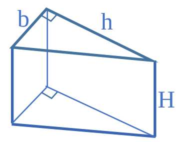
On the next page are three problems for you to solve by cutting the region into infinitesimals and then adding them up. I hope this will be affirming instead of frustrating.

Break each region into infinitesimals and then add them up. Learn from the answer to each to help you do the next. Hints is given at the bottom of this page and answers are given at the top of the next page.

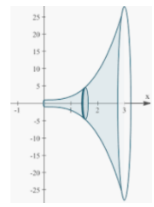
C...use integration to find the area of right triangle of base b and height h.



D. use integration to find volume of a right prism of height H with a base of a right triangle of base b and height h.



E…find the volume of x^3+1 rotated about x axis from x = 0 to 3.

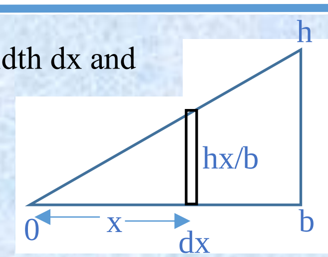


Here is a little distraction before we get to the hints.

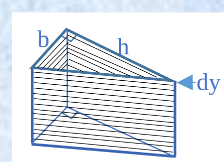
Since by definition, v is the derivative of x with respect to time and a is the derivative of v with respect to time, it follows that v is the integral of a with respect to time and x is the integral of v with respect to time. But in the case of integration we need to remember the constant of integration.

F. Integrate acceleration "a" with respect to time, $\int a\,dt$, and call the unknown constant "v_i" (velocity at beginning of the interval) . Integrate your result with respect to time and call the unknown constant "x_i". If you remember that $x - x_i = \Delta x$, then this result should look very familiar.

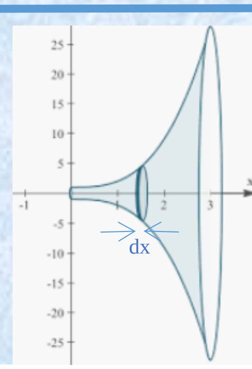
Hint A: Cut the triangle into infinitesimals of area that are rectangles of width dx and height that is a function of x which ranges from 0 to b.
There are two similar triangles and h/b =(height of rectangle)/x, thus the height of the rectangle is hx/b.
Now you should be able to set up an integral to add up all the rectangle areas (hx/b times dx) from x = 0 to x = b.



Hint B: The answer to A was that the area = ½ bh.
Cut the prism into infinitesimals of volume which are prisms with right triangle b-h as base and thickness dy.
Now you should be able to set up an integral to add up all the triangle **volumes** (½ bh times dy) from y = 0 to y = H.



Hint C: Cut the volume into infinitesimals of volume which are disks as shown of thickness dx and area $\pi r^2 = .\ \pi(x^3 + 1)^2$
Add $\pi(x^6 + 2x^3 + 1)$ times dx from x = 0 to x = 3.



C…½ bh

D. ½ bhH

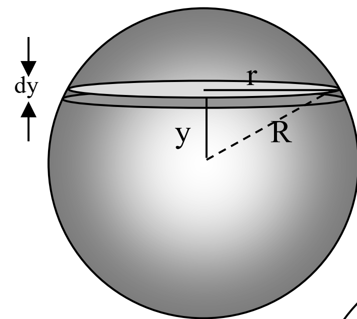
E. $\pi\left(\frac{1}{7}x^7+\frac{2}{4}x^4+1\right)\Big|_0^3 = \pi\left(\frac{2187}{7}+\frac{262}{4}+1\right)-\pi = 1108.76$

F. $x = \frac{1}{2}at^2 + v_i t + x_i$

G. Recall that work is done when a force acts through a distance. $W = F \cdot x$
Find the work done when the force given by $F(x) = 3x^2$ acts from 1 meters to 4 meters

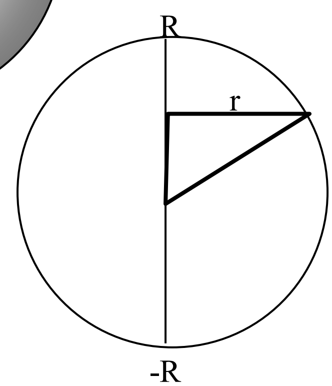
H. Find the area under the curve $v(t) = 12t^3 + 5t^2 - 8t + 3$ from 1 second to 5 seconds.

I. One way to find the volume of a sphere of radius R is to chop it into thin disks and add up their volumes. If the sphere at the right is centered at the origin and we chop it into horizontal disks of thickness dy then the volume of each disk will be $\pi r^2 dy$.
To use an integral we need to know r as a function of y (the variable with respect to which we are integrating).
For any value of y, $r = \sqrt{R^2 - y^2}$.



Evaluate this integral to find the volume:

$$\int_{-R}^{R} \pi\left(\sqrt{R^2-y^2}\right)^2 dy = \int_{-R}^{R} \pi(R^2-y^2)dy =$$



Another way is to add up spherical shells of thickness dr (like the layers of an onion)

G. 63 joules H. 1994.7 I. $\frac{4}{3}\pi R^3$

INTEGRATION BY SUBSTITUTION

Integration seems often to involve learning clever tricks that can get you to the answer.
The occasion will occur often where you must evaluate an integral like $\int \frac{x}{x^2-3}dx$
You will learn from experience that this can be accomplished by **substituting** u for $x^2 - 3$
If $u = x^2 - 3$ then du/dx = 2x or du = 2xdx
Rewrite the original integral as $\frac{1}{2}\int \frac{2x\,dx}{x^2-3}$ and now it equals $\frac{1}{2}\int \frac{du}{u}$ which I hope you recall = ½ ln(u)
Then substitute back to get that $\int \frac{x}{x^2-3}dx \quad = \frac{1}{2}\ln(x^2-3) + C$
(*if this were a definite integral, we'd want to return to "x form" before putting in the limits of integration*)

So let's practice. These problems are both worked and answered at the top of the next page. Check each as you do it (or give up on doing it) and then come back for the next problem.

J. $\int 3xsinx^2dx$

K. $\int x^4 e^{x^5} dx$

L. $\int (1+x^3)^5 x^2 dx$

But suppose in K we were instead asked to find $\int x^3 e^{x^5} dx.$ I have no idea how to proceed. I looked in standard tables. I tried googling. I had no luck! Usually, I would have to go to Ms. Reed who knows how to make Mathematica force out a solution. The point being, there are some integrals that are not nicely solved. And there are other integrals that yield to other forms of cleverness that we won't go into in AP Physics.

J. u = x^2 du = 2xdx $\frac{3}{2}\int sinx^2\, 2xdx = \frac{3}{2}\int sinu\, du =$ $\boldsymbol{\frac{3}{2}\, cosx^2}$

K. u = x^5 du = $5x^4$dx $\frac{1}{5}\int e^{x^5}\, 5x^4dx = \frac{1}{5}\int e^u\, du =$ $\boldsymbol{\frac{1}{5}e^{x^5}}$

L. u = 1+ x^3 du = $3x^2$dx $\frac{1}{3}\int (1+x^3)^5 3x^2dx = \frac{1}{3}\int (u)^5 du =$ $\boldsymbol{\frac{1}{15}(1+x^3)^5}$

DIFFERENTIAL EQUATIONS *(last topic… hurray!!!, but also the most scary… booo!!!)*

A differential equation is an equation with both the variable and its derivative among its terms. For example, a ball is falling through air, but wind resistance causes a retarding force that depends on velocity as: F_{drag} = -2v.
ΣF = ma
Weight – Drag = ma
mg – 2v = ma = m dv/dt so this is a differential equation with both v and dv/dt in its terms.

Stay with me as we go through how to solve this differential equation. First we take $\frac{dv}{dt}$ which is really the differential operator, $\frac{d}{dt}$ acting on v, but we will act as if $\frac{dv}{dt}$ was dv divided by dt and multiply both sides by dt

mg – 2v = m$\frac{dv}{dt}$ becomes (mg – 2v)dt = mdv

we need all the v's together: dt = $\frac{mdv}{mg-2v}$ if these two things are equal, then so are their integrals

$\int dt = \int \frac{mdv}{mg-2v}$ let u = mg-2v, du = -2dv

t = - ½ m ln(mg-2v)+ C wow look at us! C is constant of integration

But we want to solve for v

Subtract C and multiply by $\frac{-2}{m}$ -2(t-C)/m = ln(mg-2v) make both sides of equation exponents of e

$e^{-2(t-C)/m} = e^{-2t/m}\, e^{2C/m} =$ mg-2v now to deal with C

At time t= 0, v = v_o

Subbing these into the equation $e^{0/m}\, e^{2C/m} = e^{2(C)/m}$ = mg – $2v_o$

Letting $e^{2(C)/m}$ = mg – $2v_o$ **(mg – $2v_o$) $\boldsymbol{e^{-2t/m}}$ = mg – 2v** very scary looking, but get used to it.

You will become very experienced with working with these "first degree" differential equations in both the context of mechanics (forces) and electricity.

Second degree differential equations will also occur, but they will not involve the processes above. Instead you will just memorize the solution as shown below.

You may recall from last year's physics that elastic objects obey Hooke's Law: restoring force is proportional to deformation, F = -kx … ma = -kx but a = $\frac{dv}{dt} = \frac{d\frac{dx}{dt}}{dt} = \frac{d^2x}{dt^2}$ *(called the second derivative of x with respect to t)*

m$\frac{d^2x}{dt^2}$ = -kx which is a 2nd degree differential equation *(since there's a 2nd derivative)*

The solution we will use will be similar to x = $\cos\left(\sqrt{\frac{k}{m}}t\right)$.

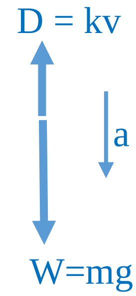
Then v = dx/dt = $-\sqrt{\frac{k}{m}}\, sin\left(\sqrt{\frac{k}{m}}t\right)$ and a = dv/dt = $\frac{d^2x}{dt^2} = -\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right)$ *(don't forget the chain rule)*

Hooke's Law: m$\frac{d^2x}{dt^2}$ = -kx

Becomes: $m\left(-\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right)\right) = -k\cos\left(\sqrt{\frac{k}{m}}t\right)$ which you can see is a true expression.

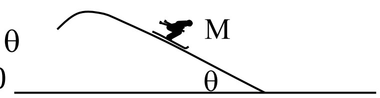
We finish this summer assignment with some problems that are very similar to those you will be doing next Fall. The first one will be completely worked out at the top of the next page. Try to just look enough to regain your confidence and then go back to trying to do it on your own… until you need another reminder.

M. A small body of mass 8.0 located near the Earth's surface falls from rest in the Earth's gravitational field. Acting on the body is a resistive force of magnitude kv, where k is 3.0 kg/s^2 and v is the speed of the body.
Free body of this scenario is shown at right
Newton's second law of motion ($\Sigma F = ma$) yields the equation $mg - kv = ma$.

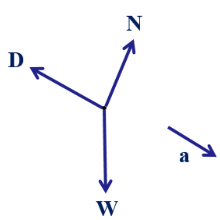


a) Write the differential equation that represents Newton's second law for this situation.
b) Integrate the differential equation once to obtain an expression for the speed v as a function of time t. Use the condition the v = 0 when t =0.
c) What is the body's speed after falling for 10 seconds?

N. A skier of mass M is skiing down a frictionless hill that makes an angle θ with the horizontal, as shown above. The skier starts from rest at time t = 0 and is subject to a velocity dependent drag force due to air resistance of the form F = -bv, where v is the velocity of the skier and b is a positive constant. Express all algebraic answers in terms of M, b, q, and fundamental constants.



The free-body of this scenario is as shown



The statements of newton's 2^{nd} that go with it are $\Sigma F_{\parallel} = ma_{\parallel}$ $\Sigma F_{\perp} = ma_{\perp}$
$mg\sin\theta - D = ma$ $mg\cos\theta - N = 0$

a) Write the differential equation that can be used to solve for the velocity of the skier as a function of time.
b) Solve the differential equation to determine the velocity of the skier as a function of time.

O. Without looking back at the previous page, starting with y = Mcos(wt),
a) What is velocity, v = dy/dt = ?
b) What is acceleration, $a = d^2y/dt^2 = \frac{d\left(\frac{dy}{dt}\right)}{dt}$ = ?
c) If y = vertical position, substitute Mcos(wt) and your answer for b into: $a = -By$
d) What is B = ?

M. a) mg – kv = ma is the same as mg – kv = m(dv/dt)

b) (mg – kv)dt = mdv

$$\frac{dt}{m} = \frac{dv}{mg-kv}$$ to get all the v's together

$$\int \frac{dt}{m} = \int \frac{dv}{mg-kv}$$ let u = mg – kv then du = -kdv

$$\int \frac{dt}{m} = -\frac{1}{k}\int \frac{du}{u}$$

$$\frac{t}{m} + C = -\frac{1}{k}\ln(mg-kv)$$ when t=0 and v=0, $C = -\frac{1}{k}\ln(mg)$

$$\frac{-kt}{m} + \ln(mg) = \ln(mg-kv)$$

$$e^{\frac{-kt}{m} + \ln(mg)} = (mg-kv)$$

$$mge^{\frac{-kt}{m}} = mg - kv$$

$$v = \frac{mg}{k}\left(1 - e^{\frac{-kt}{m}}\right)$$ which looks frightening and ugly, but you'll get used to it

c) $v = (8 \times 9.8 / 3)\left(1 - e^{\frac{-3\times 10}{8}}\right) = 25.518 = 26$ m/s

N. a) Mg sinθ -bv = Mdv/dt or dv = (Mgsinθ - bv) dt/M

b) $v = Mg\sin\theta/b\ (1 - e^{-bt/M})$

O. a) -Mwsin(wt)

b) $-Mw^2\cos(wt)$

c) $-Mw^2\cos(wt) = -B\,M\cos(wt)$

d) $B = w^2$